AN ELEMENTARY TREATISE

OF

CURVATURE.

ALSO,

A FRAGMENTARY ESSAY ON CURVES.

BY THOMAS HILL.

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PREFACE.

A TRUE system of classification is attainable only when a high degree of knowledge has been attained, because the true system must be based upon the essential laws of formation in the objects classified. Convenience demands some classification, which at first is necessarily arbitrary; and this arbitrary system may, in one sense, be called natural, since the mind, in its craving for order, and its ignorance of aught beyond what it has seen, is naturally led to form artificial order. But when, through the use of the arbitrary classification, a fuller understanding of the subject is arrived at, then the objects are newly arranged according to what appears to us to have been the law in their creation. It has been thus in botany, zoölogy, nosology, and other sciences, and will probably be thus in mathematics. The classification of curves, according to the degree of their equation in rectilinear and in polar co-ordinates, is natural to us, from our modes of inquiry and measurement; but it may not be natural to the curves; that is, it may not class them according to their laws of continuity and direction, which are the laws of their formation.

The following Treatise and Essay make, as I am well aware, very slight advances towards a better classification;

yet they will have, I think, some value. But, whatever there is of apparent novelty or real value in these speculations, they are simply the development of ideas long since published by Prof. Peirce, and frequently used by him in his investigations. Should a second and third book follow this, they will treat of "Curvature upon Curved Surfaces," and "The Curvature of Surfaces."

For her sake, through whose generosity these papers see the light, I could wish they were more valuable, and more worthy of publication.

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AN

ELEMENTARY TREATISE ON CURVATURE.

BOOK I.

OF CURVATURE IN A PLANE.

CHAP. I.

DEFINITIONS.

1. A point is a single position without dimensions.

2. A line is a position having one dimension, length.

A line may be otherwise conceived as a series of positions without dimensions, whereof each is contiguous only to two others, and they not contiguous to each other. Now, by passing in thought consecutively through this series,

3. A line may be conceived as the path of a moving point.

Although a point is immovable, yet the conception of a moving point is readily obtained from a moving body, by attending to the idea of a position in that body, — the centre, for example, of an atom or "material point."

4. The direction of a line at a given point is the direction of the motion of the supposed moving point at the given point.

5. A straight line is one whose direction is at every point the same.

6. A curve line is one whose direction continually and continuously varies.

7. A surface is a position having two dimensions, length and breadth.

8. A surface may be conceived as generated by the movement of a line, if we grant the line also may vary in form during its motion.

9. A plane is a surface generated by the motion of a straight line, whereof all the points move in the same direction, and that direction continues the same.

10. The curvature of a curve is the relative rate of change in the direction and the length of a curve.

11. The differences of two directions in a plane is called an angle.

12. As a straight line may be conceived as going in either of two directions, two straight lines crossing in a plane make four angles with each other.

13. As the difference of directions remains the same when both directions are reversed, there are but two values among the four angles made by the crossing of two straight lines. These two values are called supplements of each other.

14. When an angle and its supplement are equal, each is called a right angle.

15. A straight line passing through a point in a curve, in the same direction as the curve at that point, is called a tangent to the curve.

16. Angles made with a tangent are also made with the curve at the point of contact.

17. A straight line passing through a point in a curve, at right angles with the curve, is called a normal.

CHAP. II.

OF THE RADIUS OF CURVATURE.

1. If the curvature of a curve be constant, the curve is everywhere equally distant from a certain point, the centre.

Demonstration. — Since the curvature is constant, all parts of the curve are similar, and equal adjacent parts would coincide by super-imposition. Hence normals to the extremities of these would also coincide; whence the point of intersection of two normals is always at the same distance from the curve, and is the same point for every two normals.

2. The curve of constant curvature is, then, a re-entering curve, the simplest of all curves in its theory, and simplest for mechanical construction. It is called the circumference of a circle, or a circumference; and portions thereof are called arcs.

3. Any normal to a circumference, measured to the centre, is called a radius of the circle.

4. The angle made by two radii in a circle is proportional to the arc included by them.

For the angle made by the radii changes, since they are always normal to the arc, just as fast as the direction of the curve; and that, by the constancy of curvature, is also the rate of change in the length of the arc, which moreover is zero when the angle of the radii is zero.

5. The arc between two radii making a given angle is in a fixed ratio to the circumference.

For the arc is proportioned in a given circle to the angle, and the circumference is in all circles included between radii, making

a fixed angle equal to four right angles. Hence the arc is always the same part of a circumference that the angle between the radii is of four right angles.

6. The circumference is proportional to the radius.

Which is apparent from the fact, that, in describing a circumference, the end of the radius at the centre is stationary, and the motion of each other point proportioned to its distance from the centre. A demonstration in the more usual form can be given by dividing the circle into infinitely small triangles.

7. Hence an arc of a given angle bears a given ratio to the radius, which ratio has been adopted as the measure of the angle.

That is, an angle is measured by the length of an arc intercepted between two radii, making that angle in a circle whose radius is unity.

8. The measure of two right angles is called π; of one right angle, $\frac{1}{2}\pi$, &c. Angles are also measured by degrees; a degree being the 180th part of π.

9. The length of an arc is found by multiplying the radius by the angle, as is evident from Art. 6.

10. Hence, and from I. 10. the radius of a circle is in inverse proportion to the curvature of the circumference.

11. Hence the curvature of any point of any curve is measured by the reciprocal of the radius of the circle which would be formed were the curvature to remain constant, of the value which it has at that point. This radius is called the radius of curvature, and its extremity is called the centre of curvature.

12. The centre of curvature, in any curve, is on a curve to which the radius is tangent.

Proof. — For, since the radius of curvature is perpendicular to the curve, all increase or diminution in its length must take place at the centre, and be in the direction of the radius.

13. The curve of the preceding section is called the evo-

lute of the given curve, because, if a string be imagined to be wound tight upon it and unrolled, the unrolled portion, being tangent to it, and increasing in length exactly as fast as its point of contact moves, represents the radius of curvature, and its extremity describes the given curve.

14. The rate at which a radius of curvature is increasing or diminishing, compared with its change in direction, is measured by the radius of curvature of the evolute.

Proof. — The arc of the evolute changes as fast as the radius of the curve, and the direction of the evolute as fast as the direction of the curve; and the comparative rate of these two changes, by I. 10, II. 11, is the radius of the evolute.

CHAP. III.

SINGULAR POINTS.

1. To find a cusp of the first kind.

At such a point the radius manifestly changes its sign. Let us, then, see whether every change of sign will produce a cusp. If the change takes place by passing through zero, the radius of the evolute does not change at the same time; and therefore the evolute will be continuously convex at the point, the two values of the radius ± 0 will lie in opposite directions on the same side of the evolute, and the point will be a cusp of the first kind. If the change takes place in passing through infinity, the radius of the evolute does not change sign, and there is therefore a cusp of the first kind, usually prolonged into a straight line.

2. To find a cusp of the second kind.

At such a point it is manifest, that the radius for a change of angle assumes two values, by means of a double sign.

3. To find a point of contrary flexure.

In this case, were the function continuous, the radius of curvature would change its sign; the radius of the evolute also change its sign; and, at the same time, the angle suffer a change of direction equal to two right angles. Hence a point of contrary flexure always involves a discontinuity of the function, and generally a radius with a double sign, *discontinuing* at zero or infinity.

When there is a double sign, and the radius *passes through* zero or infinity, there are two contrary flexures, amounting to a contact of the curve with itself.

CHAP. IV.

THE CLASSIFICATION OF CURVES.

1. An equation between the radius of curvature, ϱ, and the angle it makes with a given direction, implies all the conditions of the form of the curve, though not of its position.

2. Such an equation apparently excludes, however, conjugate points.

3. The classification of these equations affords a means of classifying the curves, according to apparently natural laws.

4. The co-ordinates of this chapter may be called Peirce's Circular Co-ordinates.

FRAGMENTARY ESSAY.

FRAGMENTARY ESSAY ON CERTAIN CURVES,

TREATED ACCORDING TO THE FOREGOING PRINCIPLES.

A.

Leaving now the elementary form, which we adopted chiefly as a matter of curiosity to show the naturalness of this mode of approaching curves through curvature, let us examine some simple, particular examples. Using ϱ for the radius of curvature; ϱ', ϱ'', &c. for those of the first, second, &c. involutes; ϱ_1, ϱ_2, &c. for those of the first, second, &c. evolutes; ν for the angle made by ϱ with the axis; τ for the angle made by the curve with the axis; r for the radius vector, φ its angle with the axis; d_x for differential, and D_x for differential coefficient, relatively to x; we proceed,

1. To transform from polar to circular co-ordinates.

We have, by definition, $\varrho = D_\tau s$; and since $d\nu = d\tau$, we have $\varrho = D_\nu s$. But

$$D_\varphi s = ((D_\varphi r)^2 + r^2)^{\frac{1}{2}} \text{ and}$$

$$D_\varphi \tau = 1 + D_\varphi\,(\text{arc cot}\,(D_\varphi \log r)),$$

as is manifest from the geometry of the differentials. Putting

$$\text{arc cot}\,(D_\varphi \log r) = \varepsilon,$$

we get, by differentiation,

$$D_\varphi \varepsilon = -\sin^2\varepsilon \, D_\varphi^2 \log r. \quad \text{But}$$

$$\sin^2\varepsilon = \frac{r^2}{r^2 + (D_\varphi r)^2},$$

$$\text{and } D_\varphi^2 \log r = \frac{rD^2r - (Dr)^2}{r^2}.$$

$$\text{Whence } D_\varphi \tau = \frac{r^2 - rD^2r + 2\,(Dr)^2}{r^2 + (Dr)^2},$$

$$\text{and } \varrho = \frac{(r^2 + (Dr)^2)^{\frac{3}{2}}}{r^2 + 2\,(Dr)^2 - rD^2r}.$$

We have also

$$D_\varphi \log r = \cot\,(\tau - \varphi)\,;$$

and, by eliminating r and φ between the last two equations, and the polar equation of the curve, we obtain the equation in circular co-ordinates, putting $\tau = \nu - \frac{1}{2}\pi$.

2. To transform from rectangular to circular co-ordinates.

Since $D_x y = \tan\tau$, we have

$$D_x \tau = D_x^2 y \cos^2\tau = \frac{D^2y}{1 + (Dy)^2}, \text{ and}$$

$$\varrho = D_\tau s = \frac{(1 + (Dy)^2)^{\frac{3}{2}}}{D^2y} = \frac{\sec^3\tau}{D_x^2 y}.$$

Eliminating x and y between these equations and that of the curve, the transformation is effected, putting $\tau = \nu - \frac{1}{2}\pi$.

3. To transform from circular to rectangular co-ordinates.

Taking the axis of ν as the axis of x, we evidently have

$$x = -\int_{\nu} \varrho \sin \nu,$$

$$y = \int_{\nu} \varrho \cos \nu;$$

whence, by eliminating ν, we obtain the required equation.

B.

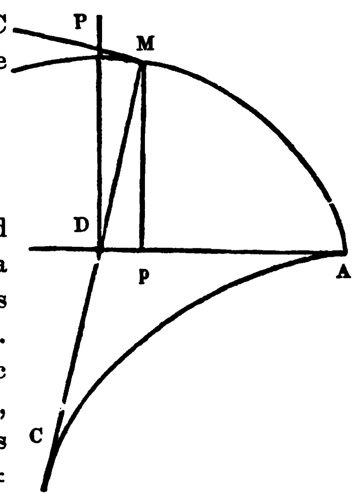
1. To investigate the curve, $\varrho = (A + B\nu^n)^m$.

If m be a fraction with an even denominator, there is a contrary flexure at the point $\nu^n = -\frac{A}{B}$.

If n be such a fraction, there is a cusp of the second kind at the origin.

If n and m be integers, the (mn)th differential coefficient is constant, and the (mn)th evolute is therefore a circle.

2. To investigate the curve, $\varrho = B \sin \nu$.



Let AM be the curve, AC the evolute, AD the axis. We have, by **A** 3,

$$Mp = \tfrac{1}{2}B \sin^2\nu;$$

whence $DM = \frac{1}{2}B \sin\nu = \frac{1}{2}\varrho$.

Draw $ADP = DMP = \frac{1}{2}\pi$, and $DP = \frac{1}{2}B$. Whence DM is a chord in a circle of the radius $\frac{1}{4}B$, tangent to the axis at D. But the differential of the arc subtended by DM is $\frac{1}{2}B\,d\nu$, and the rate at which D moves on the axis is $CD.\,d\nu.\,\mathrm{cosec}\,\nu = \frac{1}{2}B\,d\nu$. Hence, if the point M be fixed to a point in the circumference, the circle will roll on the axis without sliding; that is, the curve is a cycloid.

3. To investigate the curve, $\varrho = \mathrm{B} \cos\nu'$.

It is manifest, that, by taking an axis at right angles to that of the cycloid of **B** 2, this curve becomes identical with that, and $\nu = \nu' + \frac{1}{2}\pi$.

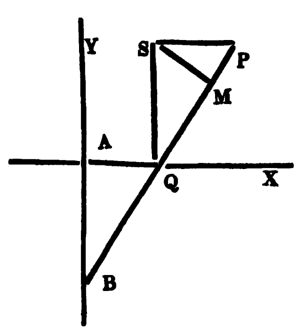
4. Hence the evolutes of a cycloid are precisely similar and parallel, the cusps of each curve resting on the vertices of the arches of its evolute.

5. To investigate the curve, $\varrho = \mathrm{A} + \mathrm{B} \sin\nu$.

Since the evolute is, by **B** 3, a simple cycloid, the curve is concentric to a similar cycloid. The cusps of the curve rest upon the arches of the evolute, at the distance A from the vertices. When $\mathrm{A} = \frac{1}{4}\mathrm{B}$, the alternate larger arches are tangent to each other at their intersection with the axis. When $\mathrm{A} = \mathrm{B}$, the smaller arches vanish, and the larger intersect at the point,

$$\sin\nu = -\frac{\mathrm{A}}{\mathrm{B}} \pm \mathrm{B}\,(\mathrm{A}^2 + \mathrm{B}^2\pi)^{\frac{1}{2}}.$$

6. To investigate the curve, $\varrho = \mathrm{B} \sin^2\nu$.



Taking the figure of **B** 2, we have

$$\mathrm{Mp} = \tfrac{1}{3}\mathrm{B} \sin^3\nu, \text{ and}$$

$$\mathrm{DM} = \tfrac{1}{3}\mathrm{B} \sin^2\nu = \tfrac{1}{3}\varrho.$$

Whence, if $\mathrm{BP} = \mathrm{B}$, $\mathrm{PQ} = \frac{1}{3}\mathrm{B}$, $\mathrm{PQX} = \nu$, and if $\mathrm{SQX} = \mathrm{PSQ} = \mathrm{SMP} = \frac{1}{2}\pi$, then M is a point in the required curve.

The rectangular co-ordinates are

$$x = (\mathrm{B}^2 - (3\mathrm{B}^2 y)^{\frac{2}{3}})^{\frac{1}{2}} —— ((\tfrac{1}{3}\mathrm{B})^{\frac{2}{3}} - y^{\frac{2}{3}})^{\frac{3}{2}}.$$

7. The curve $\varrho = A + B \sin^2 \nu$ is evidently an oval, concentric with that of the preceding article, unless A and B are of opposite signs, and B greater than A, in which case it is a series of arcs, joined in cusps at the four points where $\sin^2 \nu = -\frac{A}{B}$.

8. To investigate the curve, $\varrho = B \sin^3 \nu$.

Returning to the figure in **B** 7, we have

$$Mp = \tfrac{1}{4} B \sin^4 \nu,$$
$$\text{and } DM = \tfrac{1}{4} B \sin^3 \nu = \tfrac{1}{4} \varrho;$$

and the curve is a series of arches, whose length $= \frac{3}{4}B$, height $= \frac{1}{8}B$, and span $= \frac{3}{8}B\pi$. The evolute arches have the same span, a height of $\frac{3}{8}B$, and length $= 2B$.

9. The curve $\varrho = \frac{B}{\sin \nu}$ gives

$$Mp = B \log \sin \nu + \infty,$$
$$\text{and } Ap = -B\nu.$$

The point A is, then, at an infinite distance from the vertices. But since, when $\sin \nu$ changes sign, ϱ changes sign, and passes through precisely the same values, the curve consists of a series of arches, whose height $= \infty$, and span $= B\pi$. Of these, each alternate arch is imaginary in rectangular coordinates.

10. The curve $\varrho = \frac{B}{\sin^2 \nu}$ gives

$$Mp = \infty - \frac{B}{\sin \nu};$$

and if A′D′, at an infinite distance, is parallel to the axis,

$$D'M = \frac{B}{\sin^2} = \varrho;$$

that is, if ϱ be continued through the curve, to a distance

equal to ϱ, the termination will be on a straight line, parallel to the axis, at the distance B above the curve. It is an oval of infinite diameters, but finite curvature on two sides.

11. The curve $\varrho = \frac{B}{\sin^3 \nu}$ consists of an infinite number of arches of infinite height and span.

In rectangular co-ordinates, $y = \infty - \frac{B}{2\sin^2}$; changing the origin by subtracting $\infty - \frac{1}{2}B$, and the direction of the axis by adding π to ν, we have

$$y = \tfrac{1}{2}B\ (\text{cosec}^2 - 1) = \tfrac{1}{2}B\ \cot^2 = \tfrac{1}{2}B\ (D_x y)^2,$$

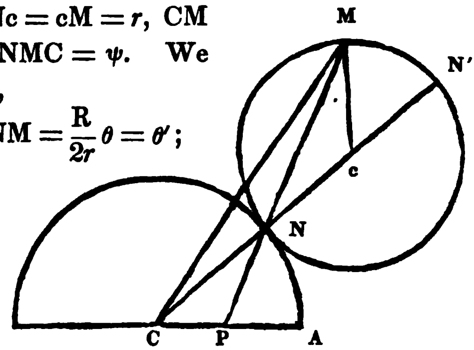
$$D_y x = B^{\frac{1}{2}}\ (2y)^{-\frac{1}{2}},\ x = (2By)^{\frac{1}{2}},\ x^2 = 2By,$$

which is the equation of a parabola.

12. It appears that, when n is an odd integer, the curve $\varrho = B \sin^n \nu$ is a series of arches, which are finite when n is positive, but have some dimensions infinite when n is negative. With n even, the curve is an oval, finite for positive values of n, but having infinite dimensions when n is negative.

For positive values of n, ϱ is divided by the axis in the proportion $\frac{1}{n+1}$. For negative values, except -1, there is a straight line, parallel to the axis, which may be called the directrix; the distance between which and the evolute, if measured on a normal to the curve, is divided by the curve in the proportion $\frac{1}{n}$.

13. To find the equation of a cycloid, when the generating circle rolls round the circumference of a circle.



Let $CN = R$, $Nc = cM = r$, $CM = x$, $NCA = \theta$, $NMC = \psi$. We have, by geometry,

$$N'cM = \frac{R}{r}\theta, \; cNM = \frac{R}{2r}\theta = \theta';$$

and, by triangle NMC, $x \sin\psi = R \sin\theta'$. (1)

There are two motions of M; by rotation about C, $= xd\theta$, and by rotation about c, $= Rd\theta$; and, as these are, by (1), proportional to the secants of their inclination to NM, this line is normal to the curve. The element, therefore,

$$ds = Rd\theta \cos\theta' + xd\theta \cos\psi. \qquad (2)$$

Also $NPA = \nu = \theta + \theta' = \theta\,(1 + \frac{R}{2r})$, and $d\theta = d\nu\,(\frac{2r}{R+2r})$.

$$\text{Whence } \varrho = D_\nu s = \frac{2Rr\cos\theta'}{R+2r} + \frac{2rx\cos\psi}{R+2r} =$$

$$\frac{2r\,(R\cos\theta' + x\cos\psi)}{R+2r}. \qquad (3)$$

But, by (1), $x = R \sin\theta' \operatorname{cosec}\psi$; and, by triangle CcM,

$$(R + r)\sin 2\theta' = x \sin(\psi + \theta'). \text{ Whence}$$

$$2\,(R + r)\sin\theta' \cos\theta' = x\,(\sin\psi\cos\theta' + \cos\psi\sin\theta') =$$

$$R \sin\theta' \cos\theta' + R \sin^2\theta' \cot\psi. \text{ Whence}$$

$$\cot\psi = \frac{R+2r}{R}\cot\theta';$$

$$x\cos\psi = R\sin\theta'\cot\psi = (R + 2r)\cos\theta';$$

which, substituted in (3), gives

$$\varrho = \frac{4r\,(\mathrm{R}+r)}{\mathrm{R}+2r}\cos\theta' = \frac{4r\,(\mathrm{R}+r)}{\mathrm{R}+2r}\cos\frac{\mathrm{R}\nu}{\mathrm{R}+2r}\,. \quad (4)$$

14. When $r=\mathrm{R}$, equation (4) becomes $\varrho = \frac{8}{3}\mathrm{R}\cos\frac{1}{3}\nu$; and, when $2r=\mathrm{R}$, $\varrho = \frac{3}{2}\mathrm{R}\cos\frac{1}{2}\nu$. In this case, $\mathrm{PNC} = \theta$, and parallel rays, impinging on the inside of AN, would form a caustic, coincident with the evolute. But the nth evolute of the epicycle is similar to the curve, differing only in the direction of the axis, and in bearing to it a ratio in linear dimensions equal to $(\frac{\mathrm{R}}{\mathrm{R}+2r})^n$. When, in equation (4), we make $\mathrm{R}=\infty$, ϱ becomes $4r\cos\nu$, which is the equation of a cycloid, as it should be.

15. To investigate the curve,

$$\varrho = \mathrm{A}\cos^2\nu + \mathrm{B}\sin^2\nu.$$

By throwing this into rectangular co-ordinates, we have

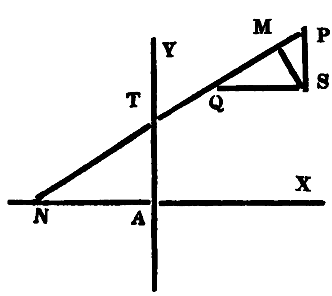
$$x = \mathrm{B}\cos\nu + \tfrac{1}{3}(\mathrm{A}-\mathrm{B})\cos^3\nu,$$

$$y = \mathrm{A}\sin\nu - \tfrac{1}{3}(\mathrm{A}-\mathrm{B})\sin^3\nu.$$

It is apparent that this curve is of similar mechanical construction to that of **B** 6.

Let $\mathrm{NP}=\mathrm{A}$, $\mathrm{PQ} = \frac{1}{3}(\mathrm{A}-\mathrm{B})$, $\mathrm{TQ}=\mathrm{B}$, and $\mathrm{TNX} = \nu$.

Draw $\mathrm{PSQ} = \mathrm{SMQ} = \frac{1}{2}\pi$, and M will be a point in the required curve.



16. The radius may be multiplied by a constant, and the preceding curve written

$$\mathrm{C}\;(\mathrm{A}\cos^2 + \mathrm{B}\sin^2) = \varrho.$$

17. In the preceding equation, let $C = AB$, and raise the binomial to the $-\frac{3}{2}$ power, and we have the radius of the ellipse, as found through rectangular co-ordinates.

18. To transform $y = (x + b)(x - b)(x - a)$ into circular co-ordinates.

We have, by differentiation,

$$-\cot\nu = 3x^2 - 2cx - b^2;$$

$$\text{whence } 3x - c = (\tfrac{1}{3}c^2 + b^2 - \cot\nu)^{\frac{1}{2}}.$$

$$\text{But } D^2y = 6x - 2c, \text{ and, by } \mathbf{A}\ 2,$$

$$\varrho = \operatorname{cosec}^3\nu\ (2\ (\tfrac{1}{3}c^2 + b^2 - \cot\nu)^{\frac{1}{2}})^{-1}.$$

The curve is then imaginary, until $\cot\nu = \frac{1}{3}c^2 + b^2$. Then it begins with $\varrho = \pm\infty$, and sweeps in both directions at once, making a contrary flexure. When $\nu = \pi$, the curve is again imaginary, and so remains until

$$\nu = \pi + \text{arc cot } (\tfrac{1}{3}c^2 + b^2),$$

when ϱ reappears $= \mp\infty$, and either describes the same curve again or one parallel to it, whose position is dependent on conditions not expressed in the equation. A third interpretation might make the curve described by the positive denominator distinct from that described by the negative, in which case a complete revolution is required for each branch of the curve.

19. Find a curve which is its own involute.

This problem was solved in Gill's "Mathematical Miscellany" for May 1839, by Prof. Peirce.

Putting $a - \frac{1}{2}\pi$ for the angle made by the axes of the two curves, we have

$$\varrho = Ae^{m\nu} + A'e^{m'\nu} +, \text{\&c.}$$

in which e is the Naperian base, and m, m', &c. the roots of the equation, $e^{ma} - m = 0$. Now, although this equation

can have but five real roots, it has an infinite number of imaginary roots when a differs from zero. The value of ϱ, therefore, may have an infinite number of arbitrary constants, A, A′, &c.; and a curve, which is its own involute, may be found so as to pass through any points whatever. A number of examples are given in the place above cited.

20. Find a curve whose fourth evolute is the curve placed parallel to its original position.

The solution of this question is given in the "Mathematical Miscellany" for November, 1839. If n be any integer, and

$$\varrho = \mathrm{A}e^{mv} + \mathrm{A}'e^{m'v} +, \&c.$$

we have A, A′, &c. arbitrary, and m, m', &c. roots of the equation, $e^{2mn\pi} = m^4$.

21. To find the ∞th involute of any curve.

Let $\boldsymbol{\Delta}$ signify an arbitrary constant; then, if the curve be geometrically continuous, so that ϱ may be developed in the form,

$$\varrho = \mathrm{A} + \mathrm{B}v + \mathrm{C}v^2 + \mathrm{D}v^3 +, \&c.$$

we shall have, for the successive involutes,

$$\varrho' = \int\varrho = \boldsymbol{\Delta} + \mathrm{A}v + \tfrac{1}{2}\mathrm{B}v^2 + \tfrac{1}{3}\mathrm{C}v^3 +, \&c.$$

$$\varrho'' = \int\varrho' = \boldsymbol{\Delta} + \boldsymbol{\Delta}v + \tfrac{1}{2}\mathrm{A}v^2 + \tfrac{1}{6}\mathrm{B}v^3 +, \&c.$$

$$\varrho''' = \int\varrho'' = \boldsymbol{\Delta} + \boldsymbol{\Delta}v + \boldsymbol{\Delta}v^2 + \tfrac{1}{6}\mathrm{A}v^3 +, \&c.$$

$$\varrho'''' = \int\varrho''' = \boldsymbol{\Delta} + \boldsymbol{\Delta}v + \boldsymbol{\Delta}v^2 + \boldsymbol{\Delta}v^3 +, \&c.$$

Thus it appears, that, by selecting arbitrary points of commencement, we may get any involute of a single branch from any curve of a single branch, by a sufficient number of evolutions; and the cycloid is not the only "mother of curves."

22. To find the length of the arc of a given curve.

Since ϱ is the differential coefficient of ϱ', it is manifestly that of the arc, and $s = \int_\nu \varrho$.

For example, $\varrho = 4\text{R} \sin\nu$, $s = -4\text{R} \cos\nu$, and the length of a branch of the cycloid is $-(-4\text{R}) - (-4\text{R}) = 8\text{R}$.

Again, the integral of **B** 13, (4), gives

$$s = \frac{4r\,(\text{R}+r)}{\text{R}} \sin\frac{\text{R}}{\text{R}+2r}\nu,$$

and the length of a branch of the epicycle is

$$\frac{8r\,(\text{R}+r)}{\text{R}}.$$